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Nearness approximation space based on axiomatic fuzzy sets ☆

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ABSTRACT

The approximation space model was originally proposed by Pawlak (1982) [19]. It was Orłowska who first observed that approximation spaces serves as a formal counterpart of perception, or observation [16, Section 2, p. 8], in which approximations provide a means of approximating one set of objects with another set of objects using the indiscernibility relation. Topology has been used to enrich the original model of an approximation space as well as more recent models of generalized approximation spaces. In this paper, an extension of the topology neighborhood based on AFS (Axiomatic Fuzzy Sets) theory is introduced, and some interesting properties are given. Furthermore, a new generalized approximation space model is established with two application examples, which can be used to deal with information tables with many category features and viewed as a multi-granulations form of nearness approximation space models.

1. Introduction

The approximation space model was originally proposed by Pawlak, which is a framework for classifying objects by means of attributes [17–19]. Pawlak introduced approximations as a means of approximating one set of objects with another set of objects using an indiscernibility relation that is based on a comparison between the feature values of objects. It is a fundamental basis for near set as well as rough set theory that the approximation of one set by another set considered in the context of approximation spaces. In recent years, considerable work on approximation spaces and their applications have been carried [2–4,16,20,22,24,26,27,33,34,36–38,40–42,44,45]. There are close ties between approximation and general topology [25,30]. For example, the approach to lower approximation and upper approximation introduced by Pawlak is closely related to the topological interior and closure operators, respectively [29, 39]. Topology is a rich source of constructs that can be used to enrich the original model of the approximation space as well as more recent models of generalized approximation spaces. Peters et al. introduced a nearness relation that can be used to determine the "nearness" of sets that are possibly disjoint [28,29]. Moreover, authors introduced several members of a family of nearness relations including weak indiscernibility relation, tolerance relation [23,25,30]. Yao discussed the general structures of approximation spaces and algebraic properties of families of subsets in finite approximation spaces [46]. Liu and Zhu [8] presented algebraic structures of the lower and upper approximations. Pei investigated algebraic structures of the collection of definable sets of approximation spaces in several rough set models [21]. Yang and Xu discussed some algebraic aspects of generalized approximation spaces, and studied algebraic properties of various families of subsets of GA-spaces under the set-inclusion order [44]. In [1,35,47,48], authors developed different model of rough sets by introducing different kinds approximation operations, respectively. Gomolinska presented an extension of a similarity-based approximation space by an additional

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Table 1
Descriptions of the information system.

	Age	Appearance		Wealth		Gender		Hair color
		Height	Weight	Salary	Estate	Male	Female	Black
x_1	21	1.69	50	0	0	1	0	5
x_2	30	1.62	52	120	200	0	1	1
x_3	27	1.80	65	100	40	1	0	3
x_4	60	1.5	63	80	324	0	1	4
x_5	45	1.71	54	145	940	1	0	2

relation understood as a relation of dissimilarity of objects, in which both positive and negative cases are taken into account in approximate reasoning about objects [4].

AFS (Axiomatic Fuzzy Sets) theory was firstly proposed by Liu in 1998 [9,10]. The AFS theory is based on the AFS algebras and AFS structure. An AFS structure is a triple (M, τ, X) which is a special family of combinatorical objects, where X is the universe of discourse, M be the set of fuzzy or crisp assertions (concepts) about features values for an object $x \in X$ (e.g., linguistic labels on assertions such as “large", “medium", “small") and τ is mathematical abstract of the complicated relations among the objects and assertions about the features values determined by the original data and facts. An AFS algebra is a family of molecular lattices generated by sets such as X, M. In essence, the AFS framework provides an effective tool to convert the information in the training examples and databases into the membership functions and their fuzzy logic operations. Recently, AFS theory has been further developed and applied to fuzzy clustering analysis [14], fuzzy decision trees [15] and concept representations [39], decision management [43], etc. About the detail properties of AFS algebras, please see [11,12].

The aim of this paper is to extend the Pawlak’s model for an approximation space and to consider the extension of generalized approximations spaces by introducing $^*EI$ algebra neighborhood. From the point of granulation, the proposed extension model can be viewed as a multi-granulations form of nearness approximation space. The article is organized as follows: In Section 2, some notations of AFS algebras are recalled. In Section 3, an extension of the original EI algebra neighborhood model, which called $^*EI$ algebra neighborhood, is introduced and the mathematical properties of $^*EI$ algebra neighborhood are discussed. In Section 4, the $^*EI$ algebra neighborhood model to be applied for the development of approximation space models. In Section 5, two application examples of the $^*EI$ algebra neighborhood model are given. Finally, Section 6 concludes the paper.

2. $^*EI$ algebra

In this section, we recall the notations and present several pertinent results of $^*EI$ algebra. The following example, which employs the information table, serves as an introductory illustration of the $^*EI$ algebra.

Example 1. Let $X = \{x_1, x_2, \ldots, x_5\}$ be a set of 5 people with feature set $F = \{f_1, f_2, \ldots, f_8\}$, and which are described by real numbers (f_1: age, f_2: height, f_3: weight, f_4: salary, f_5: estate), Boolean values (f_6: male, f_7: female) and the ordered relations (f_8: hair black). Let $M = \{m_1^1, m_2^1, m_1^2, m_2^2, m_1^3, m_2^3, m_1^4, m_2^4, m_1^5, m_2^5, m_1^6, m_1^7, m_1^8\}$, in which m_j^i is the jth assertion about f_i, m_1^i = large, m_2^i = small, $(i = 1, 2, 3, 4, 5)$, m_1^6 = male, m_1^7 = female, m_1^8 = black. Let F_i be the set of feature values on ith feature f_i, and ϕ_i be the partial function $\phi_i : X \to F_i$. For each $m \in M$ is an assertion of a feature value about an object x of the form $m_i(x)$ = ‘$\phi_i(x)$is some value$'$. For example, for object x_1, put $m_1^1(x_1)$ = ‘$\phi_1(x_1)$ is large’, $m_1^2(x_1)$ = ‘$\phi_2(x_1)$is large’, and so on. In Table 1, the number i in the “*hair color*” columns which corresponds to some $x \in X$ implies that the hair color of x is ordered ith following our perception of the color by our intuitive perception. In this ordering of objects, $x_i > x_j$ (e.g., $x_1 > x_4$) asserts the fact that $\phi_8(x_1) > \phi_8(x_4)$, i.e., the hair color of x_1 is closer to the black color than the color of hair of individual x_4.

In fact, a complex assertion about feature value of an object x may correspond to one or more assertions about feature values. For each set of features $A \subseteq M$, $\prod_{m \in A} m$ represents conjunction of the assertions about features values in A. For instance, $A = \{m_1^1, m_1^4\} \subseteq M$, $\prod_{m \in A} m = m_1^1 m_1^4$ represents a complex assertion about fuzzy feature values of f_1 and f_2 for $x \in X$, $m_1^1 m_1^4(x)$ = ‘$\phi_1(x)$ is large and ϕ_4 is large’. $\sum_{i \in I}(\prod_{m \in A_i} m)$, which is a formal sum of the assertions about some feature values, is the disjunction of the conjunctions represented by $\prod_{m \in A} m$, $A_i \subseteq M$, $i \in I$. For example, $\gamma = m_1^1 m_1^4 + m_1^1 m_1^7$ (the “+” denotes here a disjunction of the assertion about features) is a complex assertion, $\gamma(x)$ = ‘$\phi_1(x)$is large and $\phi_4(x)$ is large’ or ‘$\phi_1(x)$is large and $\phi_7(x)$ is female’. For $A_i \subseteq M$, $i \in I$, $\sum_{i \in I}(\prod_{m \in A_i} m)$ has a well-defined meaning such as the one we have discussed above.

In order to study their topological structure, we introduce $^*EI$ algebra.

Definition 1 [13]. Let M be a finite set. In general, M is a set of fuzzy or crisp assertions about feature values,

$$EM^* = \left\{ \sum_{i \in I} \left(\prod_{m \in A_i} m \right) \mid A_i \subseteq M, \quad i \in I, \ \ I \text{ is any nonempty indexing set} \right\}. \tag{1}$$

Each $\sum_{i\in I}(\prod_{m\in A_i} m)$ is an element of EM^*, where $\sum_{i\in I}$ is just a symbol meaning that element $\sum_{i\in I}(\prod_{m\in A_i} m)$ is composed of $\prod_{m\in A_i} m$.

Definition 2 [13]. Let M be a nonempty set and define a binary relation R on EM as follows: for any $\sum_{i\in I}\prod_{m\in A_i} m$, $\sum_{j\in J}\prod_{m\in B_j} m \in EM^*$, $(\sum_{i\in I}\prod_{m\in A_i} m)R(\sum_{j\in J}\prod_{m\in B_j} m) \Longleftrightarrow$ for any $A_i(i\in I)$, there exists $B_h(h\in J)$ such that $A_i \supseteq B_h$ and for any $B_j(j\in J)$, there exists $A_u(u\in I)$ such that $B_j \supseteq A_u$. It is obvious that R is an equivalence relation. The quotient set EM^*/R is denoted by EM.

Indeed, an element of EM is an equivalence class. Let $\left[\sum_{i\in I}(\prod_{m\in A_i} m)\right]_R \in EM$ be the set of all elements which are equivalent to $\sum_{i\in I}(\prod_{m\in A_i} m) \in EM^*$. In general, for any $\xi, \zeta \in EM^*$, that ξ, ζ are equivalent under R means $\xi \in [\zeta]_R$, $\zeta \in [\xi]_R$, and $[\zeta]_R = [\xi]_R$. For example, by the comparison of the semantic meanings of $\xi = m_1^1 m_2^4 + m_2^2 m_1^8$ and $\zeta = m_1^1 m_2^4 + m_2^2 m_1^8 + m_2^2 m_1^7 m_1^8$ in Example 1, one can get $[\xi]_R = [\zeta]_R$ from Definition 2.

Theorem 1 [13]. *Let M be a nonempty set. Then $(EM, \vee, \wedge)$ forms a complete lattice under the binary operations $\vee, \wedge$ defined as follows: for any $\sum_{i\in I}\prod_{m\in A_i} m$, $\sum_{j\in J}\prod_{m\in B_j} m \in EM$,*

$$\sum_{i\in I}\left(\prod_{m\in A_i} m\right) \wedge \sum_{j\in J}\left(\prod_{m\in B_j} m\right) = \sum_{k\in I\sqcup J}\left(\prod_{m\in C_k} m\right), \tag{2}$$

$$\sum_{i\in I}\left(\prod_{m\in A_i} m\right) \vee \sum_{j\in J}\left(\prod_{m\in B_j} m\right) = \sum_{i\in I, j\in J}\left(\prod_{m\in A_i\cup B_j} m\right), \tag{3}$$

where for any $k \in I \sqcup J$ (disjoin union of indexing sets I and J), $C_k = A_k$ if $k \in I$ and $C_k = B_k$ if $k \in J$. $(EM, \vee, \wedge)$ is called the $^*EI$ (expanding one set M) algebra over M.

Proof. See Appendix A.

In $^*EI$ algebra $(EM, \vee, \wedge)$, for any $\sum_{i\in I}(\prod_{m\in A_i} m)$, $\sum_{j\in J}(\prod_{m\in B_j} m) \in EM$, $\sum_{i\in I}(\prod_{m\in A_i} m) \le \sum_{j\in J}(\prod_{m\in B_j} m)$ if and only if for any $B_j(j\in J)$ there exists $A_k(k\in I)$ such that $B_j \supseteq A_k$. Notice $\emptyset$ and $\prod_{m\in M} m$ are the minimum element and maximum element of $^*EI$ algebra, respectively. That is, $^*EI$ algebra can be viewed as the opposite of EI algebra [9,11], where $*$ denotes the dual operation. □

In Example 1, let $\psi_1 = m_2^1 m_1^7 + m_1^2 m_1^3 m_1^6$, $\psi_2 = m_1^2 m_1^6 + m_1^6 m_1^8 \in EM$. By (2), (3) and Definition 1, the algebra operations of them are shown as follows:

$$\psi_2 \wedge \psi_2 = m_2^1 m_1^7 + m_1^2 m_1^6 + m_1^6 m_1^8 + m_1^2 m_1^3 m_1^6 = m_2^1 m_1^7 + m_1^2 m_1^6 + m_1^6 m_1^8,$$
$$\psi_1 \vee \psi_2 = m_2^1 m_1^2 m_1^6 m_1^7 + m_2^1 m_1^6 m_1^7 m_1^8 + m_1^2 m_1^3 m_1^6 + m_1^2 m_1^3 m_1^6 m_1^8 = m_2^1 m_1^2 m_1^6 m_1^7 + m_2^1 m_1^6 m_1^7 m_1^8 + m_1^2 m_1^3 m_1^6.$$

Definition 3 [9,11]. Let X, M be two sets and 2^M be the power set of M, $\tau : X \times X \to 2^M$. (M, τ, X) is called an AFS structure if τ satisfies the following conditions:

AX1: for any $(x_1, x_2) \in X \times X$, $\tau(x_1, x_2) \subseteq \tau(x_1, x_1)$;
AX2: for any $(x_1, x_2), (x_2, x_3) \in X \times X$, $\tau(x_1, x_2) \cap \tau(x_2, x_3) \subseteq \tau(x_1, x_3)$.

In addition, X is called universe of discourse, τ is called a structure.

Definition 4 [13]. Let ζ be any assertion about feature values. R_ζ is called a binary relation (i.e., $R_\zeta \subset X \times X$) of ζ if R_ζ satisfies: $x, y \in X$, $(x, y) \in R_\zeta \Leftrightarrow x$ belongs to ζ at some degree and the degree of x belonging to ζ is larger than or equals to that of y, or x belongs to ζ at some degree and y does not at all.

In practice, τ is defined as follows:

$$\tau(x_i, x_j) = \{m \mid m \in M, (x_i, x_j) \in R_m\} \quad \text{for any } x_i, x_j \in X,$$

It easily verifies that τ satisfies AX1, AX2 and (M, τ, X) is an AFS structure. Let $M = \{m_1^i \mid i = 1, 2, \ldots, 8\}$, according to Table 1 and the order relations of feature values, $\tau(x_4, x_5) = \{m_1^1, m_1^3, m_1^7\}$. This implies that the degree of x_4 to the assertions about features values on f_1, f_3, f_7 is superior than or equal to that of x_5.

3. Introduction of neighborhood based on $^*EI$ algebra

In this section, with the aim of applying AFS theory to approximation space, we introduce the topological neighborhood based on $^*EI$ algebra.

Definition 5 [10]**.** Let X and M be nonempty sets, and (M, τ, X) be an AFS structure. $\eta \subseteq EM$, η is a closed topology, if $\emptyset$, $\prod_{m\in M} m \in \eta$, and η is closed under finite unions ($\vee$) and arbitrary intersections ($\wedge$), and then η is called a topological molecular lattice on $^*EI$ algebra over M, denoted as (EM, η).

Definition 6 [13]**.** Let η be a topological molecular lattice on the lattice EM. If for any $\sum_{i\in I}\prod_{m\in A_i} m \in \eta$, $A_i \in \eta$ for any $i \in I$, then η is called an elementary topological molecular lattice on $^*EI$ algebra over M.

Definition 7 [13]**.** Let X and M be nonempty sets, and (M, τ, X) be an AFS structure. η is a topological molecular lattice on $^*EI$ algebra $(EM, \vee, \wedge)$ over M. For any $x \in X$, $\sum_{i\in I}\prod_{m\in A_i} m \in EM$, and $\sum_{i\in I}\prod_{m\in A_i} m \in \eta$, define $^*EI$ algebra neighborhood of x induced by $\sum_{i\in I}\prod_{m\in A_i} m \in \eta$ as follows:

$$N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x) = \left\{ y \middle| \prod_{m\in\tau(x,y)\cap\tau(y,y)} m \geq \sum_{i\in I}\prod_{m\in A_i} m \right\}, \quad \text{and} \quad N^{\tau}_{\eta}(x) = \left\{ N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x) \neq \emptyset \middle| \sum_{i\in I}\prod_{m\in A_i} m \in \eta \right\}$$

is called $^*EI$ algebra neighborhood of x induced by η.

Proposition 1. *Let X and M be sets and (M, τ, X) be AFS structure. Let η be a topological molecular lattice on $^*EI$ algebra $(EM, \vee, \wedge)$ over M. For any $x \in X$, $\alpha = \sum_{i\in I}\prod_{m\in A_i} m$, $\beta = \sum_{j\in J}\prod_{m\in B_j} m \in EM$, the following properties hold:*

(1) *If $\alpha \geq \beta$ in $^*EI$ algebra $(EM, \vee, \wedge)$, then $N^{\tau}_{\alpha}(x) \subseteq N^{\tau}_{\beta}(x)$ for any $x \in X$.*
(2) *$N^{\tau}_{\alpha}(x) \cap N^{\tau}_{\beta}(x) = N^{\tau}_{\alpha\vee\beta}(x)$ for any $x \in X$.*
(3) *$N^{\tau}_{\alpha}(x) \cup N^{\tau}_{\beta}(x) = N^{\tau}_{\alpha\wedge\beta}(x)$ for any $x \in X$.*

Proof. See Appendix B. □

Theorem 2. *Let X and M be nonempty sets, and (M, τ, X) be an AFS structure. η is an elementary topological molecular lattice on $^*EI$ algebra $(EM, \vee, \wedge)$ over M of AFS structure (M, τ, X). If*

$$B_\eta = \left\{ N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x) | x \in X, \quad \sum_{i\in I}\prod_{m\in A_i} m \in \eta \right\}$$

then B_η is a base for some topology of X.

Proof. See Appendix B. □

The topological space (X, T_η), in which B_η is a base for T_η, is called the topology induced by η.

4. Generalized approximation space model based on *$^*EI$* algebra neighborhood model

In this section, by combining generalized approximation spaces and $^*EI$ algebra neighborhood model, we propose an extension model of the approximation space, which can be viewed as a multi-granulations form of nearness approximation space proposed by Peters et al. [23,25,28,29].

Several generalizations of the classical rough set approach based on approximation spaces defined as pairs of the form (O, R), where R is the equivalence relation (called an indiscernibility relation) on a nonempty set O, have been reported in the literature (see, e.g., [4,20,28,36,38]). A generalized approximation space model can be defined by a tuple $GAS = (O, N, \nu)$, where N is a neighborhood function defined on O with values in the powerset $P(O)$ of O (i.e., $N(x)$ is a neighborhood of the object x) and ν is an overlap function defined on the Cartesian product $P(O) \times P(O)$ with values in the interval [0, 1] measuring the degree of overlap of sets. The lower GAS_* and upper GAS^* approximation operations can be defined in a GAS by (4) and (5).

$$GAS_*(X) = \{x \in O : \nu(N(x), X) = 1\}, \tag{4}$$

$$GAS^*(X) = \{x \in O : \nu(N(x), X) > 0\}. \tag{5}$$

In the standard case, $N(x)$ is equal to the equivalence class $[x]_B$ of the indiscernibility relation $Ind(B)$ for a set of features B. Usually, $N(x) = \{y \in O : xRy\}$, where R is a tolerance (similarity) relation, $R \in O \times O$, i.e., $N(x)$ is equal to the tolerance class of τ defined by x. The standard inclusion function $\nu_{SRI}(X, Y)$ is defined for $X, Y \subseteq O$, if $X \neq \emptyset$, $\nu_{SRI}(X, Y) = \frac{|X\cap Y|}{|X|}$, else $\nu_{SRI}(X, Y) = 1$. Using the standard inclusion $\nu_{SRI}(X, Y)$, (4) and (5) amounts to (6) and (7):

$$GAS_*(X) = \{x \in O : N(x) \subseteq X\}, \tag{6}$$

$$GAS^*(X) = \{x \in O : N(x) \cap X \neq \emptyset\}. \tag{7}$$

For applications, it is important to have some constructive definitions of N and ν [23,28].

One can consider another way to define $N(x)$. Usually together with a GAS, one consider some set F of formulas describing sets of objects in the universe O of the GAS defined by semantics $||\cdot||_{GAS}$, i.e., $||\alpha||_{GAS} \in O$ for any $\alpha \in F$ [23,28]. Now, one can take the neighborhood function as shown in (8):

$$N_F(x) = \{\alpha \in F : x \in ||\alpha||_{GAS}\}, \tag{8}$$

and $N(x) = \{||\alpha||_{GAS} : \alpha \in N_F(x)\}$. Hence, more general neighborhood functions having values in $P(O)$ can be defined and as a consequence different definitions of approximations are considered [23,28]. For example, one can consider the following definitions of approximation operations in GAS defined as (9) and (10).

$$GAS_*(X) = \{x \in O : \nu(Y, X) = 1, \quad \text{for some } Y \in N(x)\}, \tag{9}$$

$$GAS^*(X) = \{x \in O : \nu(Y, X) > 0, \quad \text{for any } Y \in N(x)\}. \tag{10}$$

The above generalized approximation space (GAS) model has been extended as a result of recent work on nearness of objects [23,25,28,29]. A nearness approximation space (NAS) model is a tuple:

$$NAS = (O, \mathcal{F}, \sim \mathcal{B}_r, N_r, \nu_r), \tag{11}$$

where defined using set of perceived objects O, set of probe functions $\mathcal{F}$ representing object features, indiscernibility relation $\sim \mathcal{B}_r$ defined relative to $\mathcal{B}_r \subseteq \mathcal{B} \subseteq \mathcal{F}$ family of neighborhoods N_r, and neighborhood overlap function ν_{N_r}. The subscript r denotes the cardinality of the restricted subset B_r, where we consider $\binom{|\mathcal{B}|}{r}$, i.e., $|\mathcal{B}|$ functions $\phi_i \in \mathcal{F}$ taken r at a time to define the relation $\mathcal{B}$. This relation defines a partition of O into nonempty, pairwise disjoint subsets that are equivalence (or tolerance) classes denotes by $[x]_{\mathcal{B}_r}$, such as equivalence classes $[x]_{\mathcal{B}_r} = \{x' \in O | f(x) = f(x'), \forall f \in \mathcal{B}_r\}$. These classes form a new set called the quotient set $O/\mathcal{B}_r = \{[x]_{\mathcal{B}_r} | x \in O\}$. In effect, each choice of probe functions $\mathcal{B}_r$ defines a partition $\xi_{O,\mathcal{B}}$ on a set of objects O, namely $\xi_{O,\mathcal{B}_r} = O/\sim \mathcal{B}_r$. A family of neighborhoods $N_r(\mathcal{B})$ as follows:

$$N_r(\mathcal{B}) = \{\xi_{O,\mathcal{B}_r} | \mathcal{B}_r \subseteq \mathcal{B}\}.$$

Families of neighborhoods $N_r(\mathcal{B}_r)$ contains a set of percepts. A *percept* is a byproduct of perception, i.e., something that has been observed [28]. For example, a class in $N_r(\mathcal{B})$ represents what has been perceived about objects belonging to a neighborhood, i.e., observed objects with matching probe function values. The near set approach leads to partitions of ensembles of sample objects with measurable information content and an approach to feature selection [23,25,28,29]. The near set method considers combinations of n probe functions taken r at a time in searching for those combinations of probe functions that lead to partitions of a set of objects that has the highest information content.

In the view of granular computing, an equivalence relation (or a tolerance relation) on the universe can be regarded as a granulation, and a partition (or a cover) on the universe can be regarded as a granulation space [31,32]. Notice that any features (attributes) set can induce a certain equivalence relation in a information system. Indeed, the near set method is single-granulation method. In the literatures [31], Qian and Liang extended Pawlak's single-granulation rough set model to a multi-granulations rough set model (MGRS), where the set approximations are defined by multi equivalence relations on the universe. Inspired by literature [31], this section establish an approximation space based on AFS topology, denoted AFSAS (Approximation Space based on Axiomatic Fuzzy Sets). In AFSAS, the probe function may be taken an elements in EM (Definition 2). For example, take probe function as $\alpha = m_1^1 + m_2^3$, which means that perception condition is m_1^1 or m_2^3. So, AFSAS can be viewed as a multi-granulations form of nearness approximation space.

Inspired by Peters et al. [23,28], an approximation space model based on AFS topology is a tuple

$$AFSAS = (O, \mathcal{F}, E\mathcal{B}_r, AFSN_r, \nu_r)$$

where $E\mathcal{B}_r$ denotes the set generated by set $\mathcal{B}_r$ according to Definition 2; $AFSN_r$ denotes neighborhood set defined by Definition 7. For any $X \subseteq O$, one can consider the following definitions of approximation operations in GAS:

$$GAS_*(X) = \{x \in O : \nu(Y, X) = 1, \quad \text{for some } Y \in AFSN_r(x)\}, \tag{12}$$

$$GAS^*(X) = \{x \in O : \nu(Y, X) > 0, \quad \text{for any } Y \in AFSN_r(x)\}. \tag{13}$$

5. Application examples of the $^*EI$ algebra neighborhood model

In this section, we apply $^*EI$ algebra neighborhood to approximation space and fuzzy clustering, respectively.

Example 2. As in Example 1, let assertions set $\mathcal{B} = \{m_1^1, m_1^3, m_1^6\}$, if the map $\tau_\mathcal{B} : X \times X \to 2^\mathcal{B}$ is defined as follows: for any $x, y \in X$, $\tau_\mathcal{B}(x, y) = \tau(x, y) \cap \mathcal{B}$, then $(\mathcal{B}, \tau, X)$ is an AFS structure. Let η be the topological molecular lattice generated by $\mathcal{B}$, which are elements of $^*EI$ algebra over B. $E\mathcal{B}_r = E\mathcal{B}_3 = \eta(m_1^1, m_1^3, m_1^6)$ consists of the following:

$$\alpha_1 = m_1^1 + m_1^3 + m_1^6; \quad \alpha_2 = m_1^1 + m_1^3; \quad \alpha_3 = m_1^1 + m_1^6; \quad \alpha_4 = m_1^3 + m_1^6;$$
$$\alpha_5 = m_1^1; \quad \alpha_6 = m_1^3; \quad \alpha_7 = m_1^6; \quad \alpha_8 = m_1^1 m_1^3 + m_1^1 m_1^6 + m_1^3 m_1^6;$$
$$\alpha_9 = m_1^1 m_1^3 + m_1^1 m_1^6; \quad \alpha_{10} = m_1^1 m_1^3 + m_1^3 m_1^6; \quad \alpha_{11} = m_1^1 m_1^6 + m_1^3 m_1^6;$$
$$\alpha_{12} = m_1^1 m_1^3; \quad \alpha_{13} = m_1^1 m_1^6; \quad \alpha_{14} = m_1^3 m_1^6; \quad \alpha_{15} = m_1^1 + m_1^3 m_1^6;$$
$$\alpha_{16} = m_1^3 + m_1^1 m_1^6; \quad \alpha_{17} = m_1^6 + m_1^1 m_1^3; \quad \alpha_{18} = m_1^1 m_1^3 m_1^6; \quad \alpha_{19} = \emptyset.$$

It could be easily verified that η is a topological molecular lattice. Moreover, for each α_i, $(i = 1, \ldots, 18)$, it has definitely semantic explanation, for example $\alpha_3 = m_1^1 + m_2^3$ represents the assertions about feature values "age is large or weight is small". Now we consider the base of the topology for $X = \{x_1, x_2, x_3, x_4, x_5\}$:

$$N_{\alpha_1}(x_1) = \{x_1, x_2, x_4\}; \quad N_{\alpha_2}(x_1) = \{x_1, x_2, x_4\}; \quad N_{\alpha_3}(x_1) = \{x_1\};$$
$$N_{\alpha_4}(x_1) = \{x_1, x_2, x_4\}; \quad N_{\alpha_5}(x_1) = \{x_1\}; \quad N_{\alpha_6}(x_1) = \{x_1, x_2, x_4\};$$
$$N_{\alpha_7}(x_1) = \emptyset; \quad N_{\alpha_8}(x_1) = \{x_1\}; \quad N_{\alpha_9}(x_1) = \{x_1\}; \quad N_{\alpha_{10}}(x_1) = \{x_1\};$$
$$N_{\alpha_{11}}(x_1) = \emptyset; \quad N_{\alpha_{12}}(x_1) = \{x_1\}; \quad N_{\alpha_{13}}(x_1) = \emptyset; \quad N_{\alpha_{14}}(x_1) = \emptyset;$$
$$N_{\alpha_{15}}(x_1) = \{x_1\}; \quad N_{\alpha_{16}}(x_1) = \{x_1, x_2, x_4\}; \quad N_{\alpha_{17}}(x_1) = \{x_1\};$$
$$N_{\alpha_{18}}(x_1) = \emptyset; \quad N_{\alpha_{19}}(x_1) = \{x_1, x_2, x_3, x_4, x_5\} = X,$$

where $N_{\alpha_j}(x_i)$ is an $^*EI$ algebra neighborhood of x_i generated by multi-probes function α_j, which can be viewed as a multi-granulation neighborhood and translated as all elements whose degree of belonging to assertions m are less than or equal to that of x_i under the condition α_j.

Therefore, the $^*EI$ algebra neighborhoods of x_1 induced by η is

$$N_\eta(x_1) = \{X, \{x_1, x_2, x_4\}, \{x_1\}\},$$

which can be viewed as a multi-granulation neighborhood. Similarly, we get the $^*EI$ algebra neighborhoods of x_i $(i = 2 \ldots 5)$ induced by η:

$$\begin{aligned} N_\eta(x_2) = \{&X, \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_3\}, \\ &\{x_2, x_3, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_2\}\}; \\ N_\eta(x_3) = \{&X, \{x_1, x_3, x_4\}, \{x_1, x_3\}, \{x_3\}, \{x_3, x_4\}\}; \\ N_\eta(x_4) = \{&X, \{x_4\}\}; \\ N_\eta(x_5) = \{&X, \{x_1, x_2, x_3, x_5\}, \{x_1, x_2, x_4, x_5\}, \{x_2, x_3, x_4, x_5\}, \{x_1, x_2, x_5\}, \\ &\{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}, \{x_2, x_5\}\}. \end{aligned}$$

Let $X_1 = \{x_3, x_4, x_5\}$. By (12) and (13), we can get

$$GAS_*(X_1) = \{x_3, x_4\}.$$

The lower approximation of the set X_1 with respect to η is the set all objects, which can be certain classified as X_1 using η.

$$GAS^*(X_1) = \{x_2, x_3, x_4, x_5\}.$$

The upper approximation of the set X_1 with respect to AFS topology η is the set all objects, which can be possibly classified as X_1 using η.

The topology induced by η contains more information and has well mathematical properties, although there are some limitations on dealing with huge information tables. One can takes some subset of η to deal with real problem instead of η. In what follows, we apply the $^*EI$ algebra neighborhood generated by $A \subseteq M$ $(A \in \eta)$ to empirical examples of Taiwan airfreight forwarder for the clustering.

Example 3. In [6], the authors gathered 28 strategic criteria from scholars, experts and proprietors, and selected 30 companies of airfreight forwarder in Taiwan by random selection. Using SAS (Statistical Analysis System), they obtained seven factors: Factor 1 : *Core ability*, Factor 2: *Organization management*, Factor 3: *Pricing*, Factor 4: *Competitive forces*, Factor 5: *Finance*, Factor 6: *Different advantage*, Factor 7: *Information technology*. According to the preference ratings proposed by Liang and Wang [7], it is suggested that the decision-makers utilize the linguistic rating set

$$W = \{VL, B.VL\&L, L, B.L\&M, M, B.M\&H, H, B.H\&VH, VH\},$$

Table 2
The evaluation results of five companies [13].

Co.	Factor						
	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7
x_1	M	H	H	B.H&VH	VH	L	B.M&H
x_2	H	B.L&M	M	B.M&H	H	B.M&H	VL
x_3	H	H	B.M&H	H	H	VH	B.M&H
x_4	VL	M	H	B.VL&L	H	B.L&M	M
x_5	L	M	B.H&VH	H	B.H&VH	B.VL&L	B.M&H

where *VL*: *very low*, *B.VL&L*: *between very low and low*, *L*: *low*, *B.L&M*: *between low and medium*, *M*: *medium*, *B.M&H*: *between medium and high*, *H*: *high*, *B.H&VH*: *between high and very high*, *VH*: *very high*, to assess the attention degree of subjects of companies under each of the management strategies. The decision-makers utilize the linguistic rating as above and obtain the evaluation results as Table 2. Let $X = \{x_1, \ldots, x_5\}$ and $M = \{m_1, m_2, \ldots, m_7\}$ be the assertions set about features value on features *Factor 1* to *Factor 7*, where m_i: *great attention degree of Factor i*, $i = 1, 2, \ldots, 7$. For each $m_i \in M$, we can define a binary relation R_{m_i} on X by Definition 4. (X, τ, M) is an AFS structure if τ is defined as follows: for any $x_i, x_j \in X$, $\tau(x_i, x_j) = \{m_k \in M | (x_i, x_j) \in R_{m_k}\}$.

Let $\Lambda = \{m_1, m_2, \ldots, m_7\} \subseteq M$ and η be the topological molecular lattice generated by Λ. Let $(X, \mathcal{T}_\eta)$ be the topology space on X induced by η.

Definition 8 [13]. Let X and M be finite sets. We define $D_A(x, y)$, the *distance function on the molecular A*; $d_M(x, y)$, the *molecular differential degree*; and $s_M(x, y)$, the *molecular similarity degree* in the topological space $(X, \mathcal{T}_\eta)$ as follows: for $x, y \in X, A \subseteq M, A \in \eta$,

$$D_A(x, y) = \sum_{u \in X, x \in N^\tau_{\prod_{m \in A} m}(u), y \notin N^\tau_{\prod_{m \in A} m}(u)} \left| N^\tau_{\prod_{m \in A} m}(u) \right|; \tag{14}$$

$$d_M(x, y) = \sum_{A \subseteq M, A \in \eta} (D_A(x, y) + D_A(y, x)); \tag{15}$$

$$s_M(x, y) = 1 - \frac{d_M(x, y)}{\max_{z \in X}\{d_M(z, y)\}}. \tag{16}$$

By Definition 8, one can obtain the following fuzzy similar matrix $S = (s_{ij})_{n \times n}$, $s_{ij} = s_M(x_i, x_j)$ and the following elementary differential matrix $T = (t_{ij})_{n \times n}$, $t_{ij} = d_M(x_i, x_j)$.

$$T = \begin{bmatrix} 0 & 1513 & 1175 & 1112 & 666 \\ 1513 & 0 & 638 & 1067 & 1391 \\ 1175 & 638 & 0 & 1161 & 1263 \\ 1112 & 1067 & 1161 & 0 & 918 \\ 666 & 1391 & 1263 & 918 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1.0 & 0 & 0.2234 & 0.2650 & 0.5598 \\ 0 & 1.0 & 0.5783 & 0.2948 & 0.0806 \\ 0.2234 & 0.5783 & 1.0 & 0.2327 & 0.1652 \\ 0.2650 & 0.2948 & 0.2327 & 1.0 & 0.3933 \\ 0.5598 & 0.0806 & 0.1652 & 0.3933 & 1.0 \end{bmatrix}.$$

Then, the transitive closure of similar matrix S is S^4, i.e.,

$$(S^4)^2 = S^4 = \begin{bmatrix} 1.0 & 0.2948 & 0.2948 & 0.3933 & 0.5598 \\ 0.2948 & 1.0 & 0.5783 & 0.2948 & 0.2948 \\ 0.2948 & 0.5783 & 1.0 & 0.2948 & 0.2948 \\ 0.3933 & 0.2948 & 0.2948 & 1.0 & 0.3933 \\ 0.5598 & 0.2948 & 0.2948 & 0.3933 & 1.0 \end{bmatrix}.$$

Let the threshold $\alpha = 0.5$, the clusters are $\{x_1, x_5\}$, $\{x_2, x_3\}$ and $\{x_4\}$.

Now we consider the approximations of sets $\{x_1, x_5\}$, $\{x_2, x_3\}$ and $\{x_4\}$ under AFS neighborhood generated by $A \subseteq M$, $A \in \eta$. Notice that the $^*EI$ algebra neighborhoods of x_i $(i = 1, 2, \ldots, 5)$ induced by η is

$$N_\eta(x_1) = \{\{x_{12345}\}, \{x_1, x_4, x_5\}, \{x_1, x_2, x_3, x_4\}, \{x_1, x_5\}, \{x_1, x_4\}, \{x_1\}\};$$

$$N_\eta(x_2) = \{\{x_{12345}\}, \{x_1, x_2, x_4, x_5\}, \{x_2, x_3, x_4\}, \{x_2, x_4\}, \{x_2\}\};$$

$$N_\eta(x_3) = \{\{x_{12345}\}, \{x_2, x_3\}, \{x_2, x_3, x_4, x_5\}, \{x_2, x_3, x_4\}\};$$

$$N_\eta(x_4) = \{\{x_{12345}\}, \{x_4\}, \{x_2, x_4, x_5\}, \{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_4, x_5\}, \{x_2, x_4\}, \{x_4, x_5\}, \{x_1, x_4\}\};$$

$$N_\eta(x_5) = \{\{x_{12345}\}, \{x_2, x_4, x_5\}, \{x_2, x_3, x_4, x_5\}, \{x_4, x_5\}, \{x_5\}\},$$

where $\{x_{12345}\}$ represents the set $\{x_1, x_2, x_3, x_4, x_5\}$.

Let $X_1 = \{x_1, x_5\}, X_2 = \{x_2, x_3\}, X_3 = \{x_4\}$. By (12) and (13), we can get

$$GAS_*(X_1) = \{x_1, x_5\}, \quad GAS^*(X_1) = \{x_1, x_5\}.$$
$$GAS_*(X_2) = \{x_2\}, \quad GAS^*(X_2) = \{x_2, x_3\}.$$
$$GAS_*(X_4) = \{x_4\}, \quad GAS^*(X_4) = \{x_4\}.$$

From these approximations, we can get that the clusters $X_1 = \{x_1, x_5\}$, and $X_3 = \{x_4\}$ are certain classified, while $X_2 = \{x_2, x_3\}$ is possibly classified. In [6], the authors obtained the clusters with $\alpha \in (0.590, 0.679]$: $\{x_1, x_5\}$, $\{x_2\}$, $\{x_3\}$ and $\{x_4\}$, which are similar to the results obtained in above. Furthermore, by using the FCM algorithm with Definition 8 to the data of the 30 companies shown in Appendix C, let the cluster number c be equal to 5, we obtain the clustering results:

$$\{x_2, x_3, x_6, x_7\}, \quad \{x_1, x_4, x_5, x_{10}, x_{16}, \quad x_{21}, x_{23}, x_{25}, x_{28}\}, \quad \{x_9, x_{11}, x_{13}, x_{17}, x_{19}, x_{27}\}, \quad \{x_8, x_{18}, x_{20}, x_{24}, x_{26}, x_{29}\},$$
$$\{x_{12}, x_{14}, x_{15}, x_{22}, x_{30}\}.$$

6. Conclusion

The aim of the paper is to develop an approximation space model based on AFS theory. An extension of the original *EI* algebra neighborhood model — $^*EI$ algebra neighborhood is introduced, which can be applied in the information systems with many category features. By combining approximation space models and $^*EI$ algebra neighborhood, a new approximation space model – AFSAS is proposed, which can be viewed as an extension of approximation spaces introduced by Pawlak as well as generalized approximation spaces based on the introduction of a nearness relation by Peters. Moreover, two examples about approximation of set and fuzzy clustering based on $^*EI$ algebra neighborhood and the proposed approximation space model are given. From the point of granulation, the proposed extension model can be viewed as a multi-granulations form of nearness approximation space.

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Appendix A

Theorem 3 [5]. *Let $(L, \leq)$ be a lattice, for $x, y, z \in L$, we have the follows:*

L1. $x \wedge x = x, x \vee x = x.$
L2. $x \wedge y = y \wedge x, x \vee y = y \vee x.$
L3. $x \wedge (y \wedge z) = (x \wedge y) \wedge x, x \vee (y \vee z) = (x \vee y) \vee x.$
L4. $x \wedge (x \vee y) = x = x \vee (x \wedge y).$

Theorem 4 [5]. *Let L be any set in which there are defined two binary operations $\vee$ and $\wedge$ satisfying the conditions L1–L4 of Theorem 3. Then the following assertions hold.*

(1) *For any $x, y \in L, x \wedge y = x \Leftrightarrow x \vee y = y$;*
(2) *The L is a lattice relative to the following definition of $\leq$*

$$x \leq y \Leftrightarrow x \wedge y = x$$

and that $x \vee y$ and that $x \vee y$ and $x \wedge y$ are the supremum and infimum of x and y in this lattice.

Definition 9 [5]. A partially ordered set L is called a *complete lattice* if every subset $A = \{a_i | i \in I\}$ of L has a supremum and infimum.

Proposition 2 [13]. *Let M be a non-empty set. If $\sum_{i\in I}(\prod_{m\in A_i} m) \in EM^*, A_t \subseteq A_s, t, s \in I, t \neq s$, then $\sum_{i\in I-\{s\}}\left(\prod_{m\in A_i} m\right) = \sum_{i\in I}(\prod_{m\in A_i} m)$ $\left(i.e., [\sum_{i\in I-\{s\}}(\prod_{m\in A_i} m)]_R = \left[\sum_{i\in I}(\prod_{m\in A_i} m)\right]_R\right)$.*

Proof of Theorem 1. First, we prove that $\vee, \wedge$ are binary compositions. Let $\sum_{i\in I_1}(\prod_{m\in A_{1i}} m) = \sum_{i\in I_2}(\prod_{m\in A_{2i}} m)$, $\sum_{j\in J_1}(\prod_{m\in B_{1j}} m) = \sum_{j\in J_2}(\prod_{m\in B_{2j}} m)$, hence by Definition 2, for any $A_{1i} \cup B_{1j}, i \in I_1, j \in J_1$, there exist $A_{2k}, B_{2l}, k \in I_2, l \in J_2$ such that $A_{1i} \supseteq A_{2i}$, $B_{1i} \supseteq B_{2i}$. Thus $A_{1i} \cup B_{1j} \supseteq A_{2i} \cup B_{2j}$. Similarly, for any $A_{2i} \cup B_{2j}$, $i \in I_2$, $j \in J_2$, there exist A_{1q}, B_{1e}, $q \in I_1$, $e \in J_1$, such that $A_{2i} \cup B_{21j} \supseteq A_{1i} \cup B_{1j}$. Notice that (2) can be directly verified based on Definition 2. From (3), we have

$$\sum_{i\in I_1}\left(\prod_{m\in A_{1i}} m\right) \vee \sum_{j\in J_1}\left(\prod_{m\in B_{1j}} m\right) = \sum_{k\in I_1\sqcup J_1}\left(\prod_{m\in A_{1i}\cup B_{1j}} m\right),$$
$$\sum_{i\in I_2}\left(\prod_{m\in A_{2i}} m\right) \vee \sum_{j\in J_2}\left(\prod_{m\in B_{2j}} m\right) = \sum_{k\in I_2\sqcup J_2}\left(\prod_{m\in A_{2i}\cup B_{2j}} m\right).$$

This implies that

$$\sum_{k\in I_1\sqcup J_1}\left(\prod_{m\in A_{1i}\cup B_{1j}} m\right) = \sum_{k\in I_2\sqcup J_2}\left(\prod_{m\in A_{2i}\cup B_{2j}} m\right)$$

and $\vee$ is a binary composition. Theorem 4 states that two binary compositions satisfying the conditions L1–L4 of Theorem 3 are lattice operations. For any $\sum_{i\in I}\left(\prod_{m\in A_i} m\right), \sum_{j\in J}\left(\prod_{m\in B_j} m\right), \sum_{u\in U}\left(\prod_{m\in C_u} m\right) \in EM$ we can directly verify that $\vee, \wedge$ satisfy the condition L1–L3 of Theorem 3 by definitions.

In the following, we prove that $\vee, \wedge$ satisfy the condition $L4$ of Theorem 3. By Proposition 2, we have,

$$\begin{aligned}\left(\sum_{i\in I}\left(\prod_{m\in A_i} m\right) \wedge \sum_{j\in J}\left(\prod_{m\in B_j} m\right)\right) \vee \sum_{i\in I}\left(\prod_{m\in A_i} m\right) &= \sum_{i,j\in I}\left(\prod_{m\in A_i\cup A_j} m\right) + \sum_{i,j\in I}\left(\prod_{m\in A_i\cup B_j} m\right)\\ &= \sum_{i\in I}\left(\prod_{m\in A_i} m\right) + \sum_{i,j\in I}\left(\prod_{m\in A_i\cup B_j} m\right) = \sum_{i\in I}\left(\prod_{m\in A_i} m\right)\end{aligned}$$
$$\left(\sum_{i\in I}\left(\prod_{m\in A_i} m\right) \vee \sum_{j\in J}\left(\prod_{m\in B_j} m\right)\right) \wedge \sum_{i\in I}\left(\prod_{m\in A_i} m\right) = \sum_{i\in I}\left(\prod_{m\in A_i} m\right) + \sum_{i,j\in I}\left(\prod_{m\in A_i\cup B_j} m\right) = \sum_{i\in I}\left(\prod_{m\in A_i} m\right).$$

Therefore $\vee, \wedge$ satisfy $L4$ of Theorem 3 and $(EM, \vee, \wedge)$ is a lattice.

$$\sum_{i\in I}\left(\prod_{m\in A_i} m\right) \geq \sum_{j\in J}\left(\prod_{m\in B_j} m\right) \Leftrightarrow \sum_{i\in I}\left(\prod_{m\in A_i} m\right) \vee \sum_{j\in J}\left(\prod_{m\in B_j} m\right) = \sum_{i\in I}\left(\prod_{m\in A_i} m\right)$$

if and only if for any A_i, there exist B_k, $(k \in J)$ such that $B_k \subseteq A_i$.

In the following, we prove that $(EM, \vee, \wedge)$ is a complete lattice. Let $\sum_{j\in I_i}(\prod_{m\in A_{ij}} m) \in EM$, $i \in I$, we prove that $\vee_{i\in I}(\sum_{j\in I_i}(\prod_{m\in A_{ij}} m)), \wedge_{i\in I}(\sum_{j\in I_i}(\prod_{m\in A_{ij}} m)) \in EM$. It is obvious that the following relationships are satisfied

$$\sum_{j\in I_i}\left(\prod_{m\in A_{ij}} m\right) \geq \sum_{i\in I}\sum_{j\in I_i}\left(\prod_{m\in A_{ij}} m\right), \quad \forall i \in I, \qquad \sum_{j\in I_i}\left(\prod_{m\in A_{ij}} m\right) \leq \sum_{f\in\prod_{i\in I} I_i}\left(\prod_{m\in\cup_{i\in I}A_{if(i)}} m\right), \quad \forall i \in I.$$

For $\sum_{u\in U}(\prod_{m\in B_u} m) \in EM$, if for any $i \in I$, $\sum_{j\in I_i}(\prod_{m\in A_{ij}} m) \geq \sum_{u\in U}(\prod_{m\in B_u} m)$, then for any $A_{i_0j_0}$, $i_0 \in I$, $j_0 \in I_{i_0}$, there exists $u_0 \in U$ such that $A_{i_0j_0} \supseteq B_{u_0}$. By the definition of $\wedge$, we have $\sum_{i\in I}\sum_{j\in I_i}(\prod_{m\in A_{ij}} m) \geq \sum_{u\in U}(\prod_{m\in B_u} m)$. This implies that $\sum_{i\in I}\sum_{j\in I_i}(\prod_{m\in A_{ij}} m) \in EM$ is the infimum of the set $\{\sum_{j\in I_i}(\prod_{m\in A_{ij}} m) \in EM | i \in I\}$, i.e.,

$$\bigwedge_{i\in I}\left(\sum_{j\in I_i}\left(\prod_{m\in A_{ij}} m\right)\right) = \sum_{i\in I}\sum_{j\in I_i}\left(\prod_{m\in A_{ij}} m\right).$$

For $\sum_{u\in U}\left(\prod_{m\in B_u} m\right) \in EM$, if $\sum_{j\in I_i}(\prod_{m\in A_{ij}} m) \leq \sum_{u\in U}(\prod_{m\in B_u} m)$ for any $i \in I$, then for any B_{u_0}, $u_0 \in U$, $i_0 \in I$, there exists $j_0 \in I_{i_0}$ such that $B_{u_0} \supseteq A_{i_0j_0}$. This implies that for any $u_0 \in U$, there exists $f_{u_0} \in \prod_{i\in I} I_i$ such that $f_{u_0}(i_0) = j_{i_0}$, $\forall i_0 \in I$ and $B_{u_0} \supseteq \cup_{i\in I}A_{if_{u_0}(i)}$. Therefore, by the definition of $\vee$, we have

$$\sum_{f\in\prod_{i\in I} I_i}\left(\prod_{m\in\cup_{i\in I}A_{if(i)}} m\right) \leq \sum_{u\in U}\left(\prod_{m\in B_u} m\right).$$

This implies that $\sum_{f\in\prod_{i\in I} I_i}(\prod_{m\in\cup_{i\in I}A_{if(i)}} m)$ is the supremum of the set $\{\sum_{j\in I_i}(\prod_{m\in A_{ij}} m)\in EM|i\in I\}$, i.e.,

$$\bigvee_{i\in I}\left(\sum_{j\in I_i}\left(\prod_{m\in A_{ij}} m\right)\right)=\sum_{f\in\prod_{i\in I} I_i}\left(\prod_{m\in\cup_{i\in I}A_{if(i)}} m\right).$$

By Definition 9, $(EM,\vee,\wedge)$ is a complete lattice.

Appendix B

Proof of Proposition 1

(1) Suppose $y\in N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x)$, $x\in X$. By the Definition 7, we know that there exists A_k, $k\in I$ such that $\tau(x,y)\cap\tau(y,y)\supseteq A_k$. Since $\sum_{i\in I}\prod_{m\in A_i} m\geq\sum_{j\in J}\prod_{m\in B_j} m$, then for each A_k, there exists B_l, $l\in J$ such that $\tau(x,y)\cap\tau(y,y)\supseteq A_k\supseteq B_l\Rightarrow\prod_{m\in\tau(x,y)\cap\tau(y,y)} m\geq\sum_{j\in J}\prod_{m\in B_j} m$. This implies that $y\in N^{\tau}_{\sum_{j\in J}\prod_{m\in B_j} m}(x)$. It follows $N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x)\subseteq N^{\tau}_{\sum_{j\in J}\prod_{m\in B_j} m}(x)$.

(2) For any $y\in N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x)\cap N^{\tau}_{\sum_{j\in J}\prod_{m\in B_j} m}(x)$,

$$\begin{aligned}
&y\in N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x)\cap N^{\tau}_{\sum_{j\in J}\prod_{m\in B_j} m}(x)\\
&\Leftrightarrow y\in N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x)\text{ and }y\in N^{\tau}_{\sum_{j\in J}\prod_{m\in B_j} m}(x)\\
&\Leftrightarrow \prod_{m\in\tau(x,y)\cap\tau(y,y)} m\geq\sum_{i\in I}\prod_{m\in A_i} m\text{ and }\prod_{m\in\tau(x,y)\cap\tau(y,y)} m\geq\sum_{j\in J}\prod_{m\in B_j} m\\
&\Leftrightarrow \prod_{m\in\tau(x,y)\cap\tau(y,y)} m\geq\sum_{i\in I}\prod_{m\in A_i} m\vee\sum_{j\in J}\prod_{m\in B_j} m\\
&\Leftrightarrow y\in N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m\vee\sum_{j\in J}\prod_{m\in B_j} m}(x).
\end{aligned}$$

(3) For any $y\in N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x)\cup N^{\tau}_{\sum_{j\in J}\prod_{m\in B_j} m}(x)$,

$$\begin{aligned}
&y\in N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x)\cup N^{\tau}_{\sum_{j\in J}\prod_{m\in B_j} m}(x)\\
&\Leftrightarrow y\in N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x)\text{ or }y\in N^{\tau}_{\sum_{j\in J}\prod_{m\in B_j} m}(x)\\
&\Leftrightarrow \prod_{m\in\tau(x,y)\cap\tau(y,y)} m\geq\sum_{i\in I}\prod_{m\in A_i} m\text{ or }\prod_{m\in\tau(x,y)\cap\tau(y,y)} m\geq\sum_{j\in J}\prod_{m\in B_j} m\\
&\Leftrightarrow \prod_{m\in\tau(x,y)\cap\tau(y,y)} m\geq\sum_{i\in I}\prod_{m\in A_i} m\wedge\sum_{j\in J}\prod_{m\in B_j} m\\
&\Leftrightarrow y\in N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m\wedge\sum_{j\in J}\prod_{m\in B_j} m}(x).
\end{aligned}$$

Proof of Theorem 2

Firstly, for any $x\in X$, since $\emptyset\in\eta$, hence $\tau(x,x)\supseteq\emptyset$, i.e., $x\in N^{\tau}_{\emptyset}(x)$. This implies that $X\in\cup_{N\in B_\eta}N$. Secondly, suppose $x\in X$, $U,V\in B_\eta$, and $x\in U\cap V$. We will prove that there exists $W\in B_\eta$ such that $x\in W\subseteq U\cap V$. By the hypothesis, one can get that there exist $\sum_{i\in I}\prod_{m\in A_i} m$, $\sum_{j\in J}\prod_{m\in B_j} m\in\eta$ such that $U=N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(u)$, $V=N^{\tau}_{\sum_{j\in J}\prod_{m\in B_j} m}(v)$ for some $u,v\in X$, i.e., there exist $l\in I$, $k\in J$, $\tau(u,x)\cap\tau(x,x)\supseteq A_l$ and $\tau(v,x)\cap\tau(x,x)\supseteq B_k$. By the fact $\tau(u,x)\cap\tau(x,x)\subseteq\tau(x,x)$ and $\tau(v,x)\cap\tau(x,x)\subseteq\tau(x,x)$, we have $x\in N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x)$ and $x\in N^{\tau}_{\sum_{k\in J}\prod_{m\in B_k} m}(x)$. For any $y\in N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(x)$, i.e., $\tau(x,y)\cap\tau(y,y)\supseteq A_l$, $\tau(x,y)\subseteq\tau(x,x)$ and $\tau(u,x)\cap\tau(x,y)\subseteq\tau(u,y)$ (Definition 3). It follows $\tau(u,y)\cap\tau(y,y)\supseteq\tau(u,x)\cap\tau(x,x)\cap\tau(x,y)\cap\tau(y,y)\supseteq A_l$. This fact implies that $\prod_{m\in\tau(u,y)\cap\tau(y,y)} m\geq\sum_{i\in I}\prod_{m\in A_i} m$ and $y\in N^{\tau}_{\sum_{i\in I}\prod_{m\in A_i} m}(u)$. Thus, we have $N^{\tau}_{\prod_{m\in A_i} m}(x)\subseteq U$. Similarity, $N^{\tau}_{\prod_{m\in B_k} m}(x)\subseteq V$. Since η is an elementary topological molecular lattice in the $^*EI$ algebra $(EM,\vee,\wedge)$, hence for any $A_l,B_k\in\eta$, and $A_l\vee B_k\in\eta$. By the virtue of Proposition 1, one has $W=N^{\tau}_{\prod_{m\in A_i} m}(x)\cap N^{\tau}_{\prod_{m\in B_k} m}(x)=N^{\tau}_{\prod_{m\in A_l\vee B_k} m}(x)\in B_\eta$ such that $x\in W\subseteq U\cap V$. Therefore, B_η is a base for some topology on X.

Appendix C

Table 3
Evaluate results of 30 companies [6].

Co.	Factor						
	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7
x_1	H	H	H	B.H&VH	VH	B.M&H	B.M&H
x_2	H	H	B.M&H	B.M&H	H	B.M&H	B.M&H
x_3	H	H	B.M&H	H	H	B.M&H	B.M&H
x_4	H	B.H&VH	H	B.H&VH	H	B.H&VH	B.M&H
x_5	H	B.H&VH	H	B.H&VH	B.H&VH	B.H&VH	B.M&H
x_6	H	H	B.M&H	M	B.M&H	B.M&H	B.M&H
x_7	H	H	M	B.H&VH	B.M&H	B.M&H	M
x_8	B.M&H	H	M	B.M&H	B.M&H	H	B.L& M
x_9	B.M&H	H	H	H	B.H&VH	B.M&H	M
x_{10}	H	VH	H	B.M&H	B.H&VH	H	B.M&H
x_{11}	M	H	M	H	B.M&H	B.H&VH	B.H&VH
x_{12}	VH	VH	H	B.H&VH	VH	H	M
x_{13}	B.M&H	H	B.M&H	H	B.H&VH	B.M&H	B.M&H
x_{14}	H	H	B.M&H	B.H&VH	B.H&VH	H	M
x_{15}	H	H	H	H	B.H&VH	VH	M
x_{16}	H	H	H	B.H&VH	B.H&VH	B.H&VH	B.H&VH
x_{17}	B.M&H	H	M	H	B.H&VH	B.M&H	B.M&H
x_{18}	M	H	B.M&H	B.H&VH	B.H&VH	M	M
x_{19}	B.M&H	H	B.M&H	B.M&H	VH	B.M&H	B.H&VH
x_{20}	B.M&H	M	M	H	B.H&VH	B.M&H	B.L& M
x_{21}	B.H&VH	VH	B.H&VH	B.H&VH	B.H&VH	B.H&VH	VH
x_{22}	H	B.H&VH	B.M&H	B.H&VH	B.H&VH	H	M
x_{23}	B.H&VH	VH	H	B.H&VH	H	B.M&H	B.M&H
x_{24}	H	B.M&H	M	M	B.H&VH	M	M
x_{25}	VH	VH	H	B.H&VH	B.H&VH	B.H&VH	B.M&H
x_{26}	H	M	H	B.H&VH	B.H&VH	B.H&VH	L
x_{27}	B.M&H	B.M&H	H	H	B.H&VH	H	B.M&H
x_{28}	B.H&VH	B.H&VH	B.M&H	H	B.H&VH	B.M & H	B.H&VH
x_{29}	H	B.M&H	M	H	B.M&H	B.M&H	L
x_{30}	H	B.H&VH	H	B.H&VH	H	H	M

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